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A Linear Space where Strongly Unique Elements of Best Approximation Exist

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1. INTRODUCTION

We consider a compact Hausdorff space B and denote by $C(B)$ the set of real or complex-valued continuous functions defined on B. Let V be an n-dimensional linear subspace of $C(B)$ and impose on each function $f \in C(B)$ the Chebyshev norm, viz. $\|f\| = \max\{|f(x)| : x \in B\}$.

We shall refer to these points $x \in B$ where the norm is attained, $\|f\| = |f(x)|$, as the norm points of f. As in [2] we define an H-set with respect to V, where V is spanned by the set of functions $\{g_1, g_2, ..., g_n\}$, as follows.

DEFINITION 1. A finite subset $\{x_1, ..., x_k\}$ of B is an H-set with respect to V if and only if the matrix equation

$$\begin{pmatrix} g_1(x_1) & \cdots & g_1(x_k) \\ \vdots & & \\ g_n(x_1) & \cdots & g_n(x_k) \end{pmatrix} \begin{pmatrix} l_1 \\ \vdots \\ l_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a solution $l = (l_1, ..., l_k)$ with each $l_i \neq 0$.

We then write the H-set as $[x_i, \lambda_i, e_i, k]$ with $\lambda_i = |l_i|$ and $e_i = \operatorname{sgn} l_i = l_i/\lambda_i$, and refer to the H-set and its point set $\{x_i\}$ by the same letter M; we normalize the λ_i so that $\sum \lambda_i = 1$.

The subspace V will satisfy the Haar condition if the smallest possible value of k is $n + 1$ for any H-set with respect to V. A minimal H-set is one for which no subset of the point set forms an H-set.

In [2] it was shown that Definition 1 is equivalent to the following

DEFINITION 2 (Rivlin and Shapiro [6]). $[x_i, \lambda_i, e_i, k]$ is an H-set with respect to V if and only if, for every $h \in V$,

$$\sum_{i=1}^{k} \lambda_i e_i h(x_i) = 0.$$

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DEFINITION 3 (Collatz [5]). $[x_i, \lambda_i, e_i, k]$ is an H-set with respect to V if and only if there is no function $h \in V$ such that $\operatorname{Re}[e_i h(x_i)] \geqslant 0$ for all i, with strict inequality for some i.

The usefulness of H-sets is shown by the following two theorems, see [2], concerning best Chebyshev approximation to a function $f \in C(B)$ by V.

THEOREM 1. *If a function $h_0 \in V$ can be found such that a subset of the norm points of $f - h_0$ is the point set of an H-set $[x_i, \lambda_i, e_i, k]$, and the error $R = f - h_0$ satisfies* sgn $R(x_i) = \bar{e}_i$ *for all i, then h_0 is a best Chebyshev approximation to f by V. Conversely, if h_0 is a best Chebyshev approximation to f by V, then some finite subset of the norm points, say $\{x_1, x_2, ..., x_k\}$, and the scalar values $e_i =$* sgn $\overline{f(x_i) - h_0(x_i)}$, *define an H-set $[x_i, \cdot, e_i, k]$.*

THEOREM 2. *If M is an H-set contained in the set of norm points of $f - h_1$, with h_1 a best Chebyshev approximation to f, then M is also an H-set in the set of norm points of $f - h_2$, where h_2 is any other best Chebyshev approximation to f by V.*

We define strong unicity as in [7, p. 36]:

DEFINITION 4. $h_0 \in V$ is a strongly unique element of best approximation to $f \in C(B)$ if there exists a constant r, $0 < r \leqslant 1$, depending only on f and V such that for every $h \in V$,

$$\|f - h\| \geqslant \|f - h_0\| + r\|h_0 - h\|.$$

If V satisfies the Haar condition, then uniqueness of best uniform approximation follows and also strong unicity, see [4, p. 80]. Analogous results are known for non-linear families satisfying the local Haar condition, see [1]. However, when the Haar condition is not satisfied, uniqueness is not guaranteed. In this paper we construct from an arbitrary linear subspace V a set of equivalence classes giving rise to best approximation which is unique and satisfies the strong unicity criterion.

Let $[C(B), M]$ be the set of all subsets of $C(B)$ the restriction of whose elements to an H-set $M = [x_i, \lambda_i, e_i, k]$ with respect to V are all equal. $[V, M]$ is similarly defined. Both are made linear spaces in the obvious way. $[f]$ denotes an element of one of them containing f.

From Definition 1 and the natural linear map T of V onto $[V, M]$, given by $Th = [h]$, we see that the linear space $[V, M]$ will have as dimension the rank of the matrix $g_i(x_j)$ in Definition 1. We define a norm on $[C(B), M]$ by $\|[f]\|_0 = \max_i |f(x_i)|$, so that we can speak on best approximation to $[f]$ by $[V, M]$.

Remark. If $h \in V$ is a best Chebyshev approximation to f by V and we construct the space $[C(B), M]$ with M an H-set which is a subset of the set of norm points of $f - h$, then all the best Chebyshev approximations to f by V are contained in $[h]$, as M is contained in the set of norm points for each of these best approximations, see Theorems 1 and 2 above.

2. Unicity Theorems

For an H-set M the existence of a best approximation to each $[f]$ by $[V, M]$ is guaranteed, see [4, p. 20]. We consider here the question of uniqueness.

Theorem 3. *If h is a best Chebyshev approximation to f by V and M is an H-set $[x_i, \lambda_i, e_i, k]$ in the set of norm points of $f - h$, then $[h]$ is a best approximation to $[f]$ by $[V, M]$ and* $\inf\{\|[f] - [h']\|_0 : [h'] \in [V, M]\} = \|f - h\|$.

Proof. We first prove that $\|f - h\| = \|[f] - [h]\|_0$.
From Theorem 1, if $f_1 \in [f]$ and $h_1 \in [h]$,

$$|f_1(x_i) - h_1(x_i)| = |f(x_i) - h(x_i)| = \|f - h\| \qquad \text{for all } i$$

because h is a best Chebyshev approximation to f by V; thus

$$\|[f] - [h]\|_0 = \max_i |f(x_i) - h(x_i)| = \|f - h\|.$$

Now let us assume that $[h]$ is not a best approximation to $[f]$ by $[V, M]$, so there is an $[h_1]$ such that

$$\|[f] - [h_1]\|_0 < \|[f] - [h]\|_0;$$

hence for all i,

$$\mathrm{Re}[e_i(f(x_i) - h_1(x_i))] < \|f - h\|,$$

i.e.,

$$\begin{aligned}\mathrm{Re}[e_i(f(x_i) - h(x_i) + h(x_i) - h_1(x_i))] \\ < \|f - h\| = e_i(f(x_i) - h(x_i)) \text{ from Theorem 1,}\end{aligned}$$

giving

$$\mathrm{Re}[e_i(h(x_i) - h_1(x_i))] < 0 \qquad \text{for all } i.$$

This contradicts Definition 3, as M is an H-set with respect to V; hence the result.

THEOREM 4. *A best approximation to* $[f]$ *by* $[V, M]$ *is unique.*

Proof. Let $[h]$ be as in Theorem 3. Let $[h_1]$ be some best approximation to $[f]$ by $[V, M]$ .Then we have for all i,

$$\begin{aligned}\mathrm{Re}[e_i(f(x_i) - h_1(x_i))] &\leqslant \|[f] - [h_1]\|_0 \\ &\leqslant \|[f] - [h]\|_0 \\ &= \|f - h\| \\ &= e_i(f(x_i) - h(x_i)).\end{aligned}$$

Thus $\mathrm{Re}[e_i(h_1(x_i) - h(x_i))] \geqslant 0$ for all i. As $[x, \lambda_i, e_i, k]$ is an H-set with respect to V, strict inequality can occur for no i. Hence $h_1(x_i) = h(x_i)$ for all i, that is $[h_1] = [h]$.

We now show that best approximation is strongly unique.

THEOREM 5. *Let h be a best Chebyshev approximation to f by V and let M be an H-set $[x_i, \lambda_i, e_i, k]$, a subset of the set of norm points. For any g in V,*

$$\|[f] - [h]\|_0 \geqslant (\|[f] - [h]\|_0^2 + r\,\|[g] - [h]\|_0^2)^{1/2}$$

which reduces in the real case to

$$\|[f] - [g]\|_0 \geqslant \|[f] - [h]\|_0 + s\|[g] - [h]\|_0$$

where r, s are positive constants depending on f and V only.

Proof. Let $\rho = \|f - h\|$; from Theorem 1 and Definition 2:

$$f(x_i) - h(x_i) = \rho \bar{e}_i \tag{1}$$

and for every $g \in V$,

$$-\lambda_j e_j g(x_j) = \sum_{\substack{i=1\\ i\neq j}}^{k} \lambda_i e_i g(x_i), \tag{2}$$

where each $\lambda_i > 0$ and $\sum_{i=1}^{k} \lambda_i = 1$.

For such g, set

$$\|[f] - [h - g]\|_0 - \|[f] - [h]\|_0 = m \quad (\geqslant 0).$$

Then

$$\|[f] - [h - g]\|_0 = \rho + m$$

and for all i,

$$|f(x_i) - h(x_i) + g(x_i)| \leqslant \rho + m. \tag{3}$$

Thus from (1) we get

$$e_i(f(x_i) - h(x_i) + g(x_i)) = \rho + e_i g(x_i);$$

hence

$$\operatorname{Re}[e_i g(x_i))] \leqslant m \qquad \text{for all } i, \tag{4}$$

and from (3),

$$2\rho \operatorname{Re}[e_i g(x_i)] + |g(x_i)|^2 \leqslant m^2 + 2\rho m \qquad \text{for all } i. \tag{5}$$

Setting $p = (\min_i \lambda_i)^{-1} > 1$, we have from (2),

$$|\operatorname{Re}[e_i g(x_i)]| \leqslant \rho m \qquad \text{for all } i,$$

and from (4) and (5),

$$|g(x_i)|^2 \leqslant m^2 + 2(1 + p)\,\rho m.$$

Hence

$$\|[g]\|_0^2 \leqslant m^2 + 2(1 + p)\,\rho m.$$

Solving this inequality gives

$$\begin{aligned} m &\geqslant -(1 + p)\,\rho + ((1 + p)^2 \rho^2 + \|[g]\|_0^2)^{1/2} \\ &\geqslant -\rho + (\rho^2 + (\|[g]\|_0/(1 + p)^2))^{1/2}. \end{aligned}$$

Using the definitions of m and ρ and letting $(1 + p)^2 = r^{-1}$, we get

$$\|[f] + [h - g]\|_0 \geqslant (\|[f] - [h]\|_0^2 + r\,\|[g]\|_0^2)^{1/2}.$$

As V is a linear space, we can write this as

$$\|[f] - [g]\|_0 \geqslant (\|[f] - [h]\|_0^2 + r\,\|[h] - [g]\|_0^2)^{1/2}.$$

In the real case we use the inequality $|e_i g(x_i)| \leqslant \rho m$ which gives for (6) $|g(x_i)| \leqslant pm$; solving (6) as before we get, with $s = p^{-1}$,

$$\|[f] - [g]\|_0 \geqslant \|[f] - [h]\|_0 + s\|[h] - [g]\|_0 .$$

We note that p, and hence r and s, depend on V and f but not on h.

COROLLARY. *If in Theorems* 4 *and* 5, *M is a minimal H-set with $k = n + 1$, then h is the unique best uniform approximation to f by V, and also, for every $g \in V$, we have*

$$\|f - g\| \geqslant (\|f - h\|^2 + r\,\|h - g\|^2)^{1/2}$$

which reduces in the real case to

$$\|f - g\| \geqslant \|f - h\| + s\|h - g\|,$$

where r and s depend only on V and f.

Proof. The matrix $g_i(x_j)$ in Definition 1 has rank n, so every element $[h]$ in $[V, M]$ consists of a single element and $[V, M]$ is essentially V. Thus uniqueness follows from Theorem 4. For the norms we have the following relationships:

$$\|f\| \geqslant \|[f]\|_0,$$
$$\|f - h\| = \|[f] - [h]\|_0 = \rho,$$

and, for some $t > 0$,

$$t\|h - g\| \leqslant \|[h] - [g]\|_0,$$

as both norms define the same topology for V.

Using these inequalities the strong unicity follows.

If now V satisfies the Haar condition, then every minimal H-set has $n + 1$ points; thus, from the last corollary, uniqueness of best approximation and strong unicity follow directly.

EXAMPLE. In [3] it has been shown that all best uniform approximations to xyz by polynomials in the three variables x, y, z of degree 2 or less, on the unit ball, are given by $k(x^2 + y^2 + z^2 - 1)$ with $|k| \leqslant 27^{-1/2}$ the error being $27^{-1/2}$. The H-set M associated with these best approximations consists of the 8 points $(\pm 3^{-1/2}, \pm 3^{-1/2}, \pm 3^{-1/2})$ with e_i equal to the product of the corresponding signs and each $\lambda_i = 1/8$.

The best approximation to $[xyz]$ will be the set of functions of the form $\alpha(x^2 + y^2 + z^2 - 1)$. The value of p in Theorem 5 is 8 so that $s = 1/8$ and for every $[g] \in [V, M]$,

$$\|[xyz] - [g]\|_0 \geqslant 27^{-1/2} + (\|[g] - k(x^2 + y^2 + z^2 - 1)\|_0/8).$$

As $x^2 + y^2 + z^2 = 1$ on the H-set, this reduces to

$$\|[xyz] - [g]\|_0 \geqslant 27^{-1/2} + (\|[g]\|_0/8).$$

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